

Relative Locality in Curved Space-time

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In this paper we construct the action describing dynamics of the particle moving in curved spacetime, with a non-trivial momentum space geometry. Curved momentum space is the core feature of theories where relative locality effects are presents. So far aspects of nonlinearities in momentum space have been studied only for flat or constantly expanding (De Sitter) spacetimes [1], relying on the their maximally symmetric nature. The extension of curved momentum space frameworks to arbitrary spacetime geometries could be relevant for the opportunities to test Planck-scale curvature/deformation of particles momentum space. As a first example of this construction we describe the particle with kappa-Poincaré momentum space on a circular orbit in Schwarzschild spacetime, where the contributes of momentum space curvature turn out to be negligible. The analysis of this problem relies crucially on the solution of the soccer ball problem.

It is widely expected that quantum gravity effects are characterized by the presence of an energy scale, usually identified with the Planck mass $M_{Pl} = \sqrt{G/\hbar} \sim 10^{19}$ GeV, and the associated length scale, the Planck length $\ell_{Pl} = \sqrt{G\hbar} \sim 10^{-35}$ m. Unfortunately, processes with characteristic energy and length of order of the Planck scales are far beyond reach of the current, and foreseeable, technologies and therefore the question arises as to if there could be some traces of quantum gravity effects observable with the current, or near future experiments.

As advocated in [2] such an opportunity might be offered by a possible semiclassical, weak gravity regime of quantum gravity, in which both G and \hbar are very small, while their ratio remains finite. In such a regime the dynamics of particles and fields depends on a mass scale κ , which in the case of elementary systems could be identified with the Planck mass. Further it is argued that the presence of the mass scale has an effect similar to the one well known in 2+1 gravity, namely that the momentum space becomes ‘curved’, i.e., starts having a non-trivial geometry, with the characteristic scale κ . See [3] for the up to date review.

Since the non-trivial geometry of momentum space almost inevitably leads to the conflict with

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absolute spacetime locality, the class of theories with curved momentum space are characterized also by the necessary controllable relaxation of the absolute locality [4], [5], resulting in the so-called relative locality principle [2], [6].

Although the relative locality principle, as originally devised, arises in the limit $G \rightarrow 0$ and therefore seems to force us to restrict our attention to the flat Minkowski space with vanishing gravity, it is possible to argue that one can still consider the models of particles and fields having curved momentum space and evolving in a nontrivially curved spacetime. First, even if in the limiting procedure leading to the curved momentum space the Newton's constant is assumed to be small, the physical effect of classical gravity, which are of order of GM , where M is the mass/energy of the source, might still be not negligible. Secondly, as in 2+1 gravity, it could be that the curvature of momentum space results from the 'topological' sector of gravity (see [3] for discussion and references) which is superimposed on the dynamical gravity sector which exhibits itself in the spacetime curvature.

The theory of motion of particles in de Sitter spacetime with non linearities in momentum space has been discussed in [1]. In the analysis reported there, the constant expansion rate is introduced, in a theory with deformed relativistic symmetries [7], [8], as a third observer-invariant scale (besides c and κ ; see also [9].)

In this paper we present the general construction of the action of a relativistic particle moving in spacetime of an arbitrary geometry encapsulated in the spacetime tetrad $e_\mu^a(x)$ with curved momentum space characterized by the tetrad $E_a^\alpha(p)$. One can show that our construction is compatible with the examples of relative locality studies present in the literature (e.g. [4], [5], [1]), as well as with the analyses based on the principle of relative locality (e.g. [2], [10], [11]).

Let us start our construction with the standard action of relativistic particle moving in flat Minkowski space. It reads

$$S^0 = \int d\tau \frac{dx^a}{d\tau} p_a - N(p^a p_a + m^2), \quad (1)$$

where N is the Lagrange multiplier enforcing the mass shell condition and we raise and lower indices in spacetime and momentum space with Minkowski metric tensor η_{ab} . In fact N is also a gauge field associated with the local reparametrization invariance on the worldline $\tau \rightarrow \tau'(\tau)$ and therefore it can be gauge fixed to an appropriate (usually constant) value. The equations of motions following from (1) are

$$\dot{p} = 0, \quad \dot{x}^a = 2N p^a, \quad p^2 = -m^2, \quad (2)$$

and to get the standard velocity-momentum relation we gauge fix $N = 1/2m$.

The action (1) can be generalized so as to describe the particle moving in curved spacetime or having curved momentum space. In the first case we introduce the spacetime tetrad $e_\mu^a(x)$ that maps the tangent space of the curved spacetime manifold to an ambient Minkowski space. The action has the form

$$S^E = \int d\tau \frac{dx^\mu}{d\tau} e_\mu^a(x) p_a - N(p^a p_a + m^2). \quad (3)$$

Since this action is a bit unusual let us stop here to check that it leads to the geodesic equation as the standard action $\int d\tau \sqrt{|g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu|}$ does. The equations of motion following from (3) are (for N gauge fixed to $1/2m$, as before)

$$\dot{x}^\mu e_\mu^a = \frac{1}{m} p^a, \quad (e_{\nu,\mu}^a - e_{\mu,\nu}^a) \dot{x}^\nu p_a - e_\mu^a \dot{p}_a = 0, \quad p^2 + m^2 = 0, \quad (4)$$

Substituting the momentum from the first equation to the second, after straightforward algebra one derives the geodesic equation.

Turning things around we can write down the action for a particle with curved momentum space [2], [3]

$$S^\kappa = - \int d\tau x^a E_a^\alpha(p) \frac{dp_\alpha}{d\tau} + N(\mathcal{C}(p) + m^2), \quad (5)$$

where the function $\mathcal{C}(p)$ in the mass shell relation is defined to be the geodesic distance from the momentum space origin to the point with coordinates p_α . The equations of motion following from (5) are

$$\dot{x}^a E_a^\alpha = N \frac{\partial \mathcal{C}(p)}{\partial p_\alpha}, \quad \dot{p}_\alpha = 0, \quad \mathcal{C}(p) + m^2 = 0. \quad (6)$$

where in the first equations we omitted the terms proportional to \dot{p}_α .

Before making the decisive step, formulating the action which combines (3) and (5) let us notice that for curved spacetime the positions x^μ have the upper index (chosen from the middle of Greek alphabet μ, ν, \dots), while for curved momentum space the momenta p_α have the lower index (chosen from the beginning of Greek alphabet α, β, \dots). Therefore the spacetime tetrad mapping from the tangent space of spacetime into the ambient Minkowski space looks like $e_\mu^a(x)$, while the momentum space one, mapping from the tangent space of the momentum space into this same ambient Minkowski space has the form $E_a^\alpha(p)$.

It follows that the only action for the particle with curved both spacetime and momentum space, which reduces to (3) or (5) when spacetime or momentum space are flat i.e., $e_\mu^a(x) = \delta_\mu^a$ or

$E_a^\alpha(p) = \delta_a^\alpha$, respectively has the form

$$S^{E\kappa} = - \int d\tau x^\mu E_a^\alpha(p) \frac{d}{d\tau} (e_\mu^a(x) p_\alpha) + N(\mathcal{C}(p) + m^2). \quad (7)$$

The equations of motion following from (7) have the form

$$E_a^\alpha \frac{d}{ds} (e_\mu^a p_\alpha) = p_\alpha e_{\nu,\mu}^a \frac{d}{ds} (x^\nu E_a^\alpha), \quad (8)$$

$$e_\mu^a \frac{d}{ds} (x^\mu E_a^\alpha) = x^\mu E_a^{\beta,\alpha} \frac{d}{ds} (e_\mu^a p_\beta) + N \frac{\partial \mathcal{C}(p)}{\partial p_\alpha}, \quad (9)$$

$$\mathcal{C}(p) = -m^2. \quad (10)$$

To see these equations in action let us consider a simple example of a body on a circular orbit in Schwarzschild geometry, with κ -Poincaré momentum space geometry [12]. In this case (see e.g., [3] for discussion and references, [10] for an explicit analysis) the non-vanishing components of the momentum space tetrad are

$$E_{(0)}^0 = 1, \quad E_{(j)}^i = e^{p_0/\kappa} \delta_i^j, \quad i, j = 1, 2, 3, \quad (11)$$

where we use parentheses (\cdot) to denote ‘flat’ indices, and the modified on-shell relation has the form

$$\cosh \frac{p_0}{\kappa} - \frac{\mathbf{p}^2}{2\kappa^2} e^{p_0/\kappa} = \cosh \frac{m}{\kappa}. \quad (12)$$

We use the standard Schwarzschild spherical spacetime coordinates (t, r, θ, ϕ) in which the non-vanishing components of the tetrad are

$$e_0^{(0)} = f(r), \quad e_1^{(1)} = f(r)^{-1}, \quad e_2^{(2)} = r, \quad e_3^{(3)} = r \sin \theta. \quad (13)$$

with $f(r) = \sqrt{1 - 2GM/r}$.

We will be interested only in the circular, planar orbit, which means that we will solve eqs. (8)–(13) supplemented by the condition $\dot{\theta} = \dot{r} = 0$. Taking these into account from (8) we get

$$\begin{aligned} \dot{p}_0 &= \dot{p}_2 = \dot{p}_3 = 0, \\ e^{p_0/\kappa} f^{-1} \dot{p}_1 &= f' p_0 \dot{t} + e^{p_0/\kappa} p_3 \dot{\phi}, \end{aligned} \quad (14)$$

while from (9)

$$\begin{aligned} f \dot{t} &= \frac{1}{\kappa} e^{p_0/\kappa} r f^{-1} \dot{p}_1 + N \frac{\partial \mathcal{C}(p)}{\partial p_0}, \\ \dot{\phi} &= \frac{e^{-p_0/\kappa}}{r} N \frac{\partial \mathcal{C}(p)}{\partial p_3}, \\ \frac{\partial \mathcal{C}(p)}{\partial p_1} &= \frac{\partial \mathcal{C}(p)}{\partial p_2} = 0. \end{aligned} \quad (15)$$

Since $\partial \mathcal{C}(p) / \partial p_i \sim p_i$ this last equation tells that $p_1 = p_2 = 0$.

Solving for $\dot{\phi}$ and \dot{t} we obtain an equation for components of momentum

$$p_3 \frac{\partial \mathcal{C}(p)}{\partial p_3} = -r f' f^{-1} p_0 \frac{\partial \mathcal{C}(p)}{\partial p_0}. \quad (16)$$

We expect that for physical bodies on the orbit in question we can safely expand \mathcal{C} in (12) to the next-to-leading order, to wit

$$\mathcal{C}(p) = 1 + \frac{p_0^2}{2\kappa^2} - \frac{\mathbf{p}^2}{2\kappa^2} \left(1 + \frac{p_0}{\kappa}\right) + \dots$$

so that eq. (16) takes the form

$$p_3^2 = r f' f^{-1} p_0^2 \left(1 - \frac{1}{\kappa} p_0 \left(1 + \frac{1}{2} r f' f^{-1}\right)\right). \quad (17)$$

We can now derive the first order in $1/\kappa$ corrections to the angular velocity. Substituting Eq. (17) in the second of Eqs. (14), one finds

$$\frac{d\phi}{dt} = \sqrt{\frac{f f'}{r}} \left(1 - \frac{1}{2\kappa} p_0 \left(1 - \frac{1}{2} r f' f^{-1}\right)\right). \quad (18)$$

As it is well known from previous studies on relative locality, if the Poisson brackets between spacetime coordinates and momenta depend on momenta, one expects the description of distant events to be affected by relative locality effects, in the form of “misleading inferences”: the coincidence of distant events, when expressed in spacetime coordinates which have Poisson brackets with momenta which depend on momenta, is not objective (see e.g. [5], [1]). In that case the equations of motion are not sufficient for establishing the physical motion of particles, but one has to consider also the non trivial properties of translations [5], [1].

We notice however that from the action (5), it is straightforward to define the canonical variables

$$\begin{aligned} P_A &= (E, P_r, P_\theta, L) = (f p_0, f^{-1} p_1, r p_2, r p_3), \\ X^A &= (T, R, \Theta, \Phi) = \left(t, e^{p_0/\kappa} r, e^{p_0/\kappa} \theta, e^{p_0/\kappa} \phi\right), \end{aligned} \quad (19)$$

such that $\{P_A, X^B\} = \delta_A^B$. Notice that E and L are respectively the particle’s energy and angular momentum, which are conserved even relaxing the circular orbits condition ($\dot{r} = 0$).

Guided by the intuition gained for the cases in which spacetime translations are still isometries, as for Minkowski or De Sitter spacetime metrics [5], [1], we could argue that canonical coordinates play a special role in relative locality frameworks, as the coordinates in which the equations of motion describe objectively the trajectories of distant particles, even for more general spacetime

metrics, where no notion of spacetime translations as a symmetry of the metric is available. Postponing a more detailed investigation of the role of relative locality inferences in curved spacetime to a following work, we nonetheless present the equation for the angular velocity also in the canonical coordinates (19). Substituting the canonical variables (19) into Eq. (18), one finds the angular velocity

$$\frac{d\Phi}{dT} = \sqrt{\frac{\delta(R)}{R^2}} \left(1 + \frac{E}{4\kappa} \frac{2 - 3\delta(R)}{(1 - 2\delta(R))^{3/2}} \right), \quad (20)$$

where $\delta(R)$ is the adimensional quantity $\delta(R) = M(R)G/c^2R$, with $M(R)$ the mass interior¹ to radius R .

Let us estimate the magnitude of the κ correction term in (20). Clearly, if the body in question is a planet in the planetary system or a star in the binary system or the galaxy, if we take $\kappa \sim M_{Pl}$ the correction E/κ will be enormously large. This is nothing but the ‘soccer ball problem’ (see [13] and references therein), and to get the right size of the correction term we must invoke the way this apparent paradox is solved. Namely, it is observed that the interactions of a composite body (which, no doubts, the planet or the star is) are governed by the deformation scale κ_{eff} which is N times bigger than the deformation scale for the constituents κ_{const} , with N being the number of constituents. Therefore κ in (20) is the effective value of the deformation parameter, which equals, for example $\kappa = N_{atoms} \kappa_{atom}$, with κ_{atom} being the deformation parameter at the atomic scale. Then, since we can safely assume in the present context that the moving body is non-relativistic $E \sim N_{atoms} m_{atom}$. Therefore the ratio p_0/κ in (20) is of order of m_{atom}/κ_{atom} , which, in the best case $\kappa_{atom} \sim M_{Pl}$ is of order of 10^{-19} , but might be few orders of magnitude smaller than that.

We could ask if the terms multiplying the κ correction in (20) would act as amplifier for the smallness of the effect. Taking $M_{tot} \sim 10^{12} M_\odot$, which is a typical order of magnitude for large mass galaxies, and $R \sim 10^3 \text{pc}$, we get $\delta(R) \sim 10^{-7}$. In view of this the corrections to the equation (20) resulting from the non-trivial momentum space geometry, in the non-relativistic regime are negligibly small.

In this paper we presented the theory of motion of a relativistic particle with curved momentum space traveling in a spacetime of an arbitrary geometry. There are many interesting physical situations where this theory can be used. The first is the cosmological setup, with spacetime being of the form of de Sitter or a general Friedmann-Robertson-Walker, the second the black

¹ To derive the relation between $\delta(r)$ and $\delta(R)$ we have assumed spherical mass distribution, with constant average density, so that $M(r)$ scales as r^3 , i.e. $M(r) = (M_{tot}/R_{tot}^3)r^3 = e^{-3p_0/\kappa}(M_{tot}/R_{tot}^3)R^3 = e^{-3p_0/\kappa}M(R)$.

hole geometry and/or Rindler space. In both cases it is expected that the remnants of quantum gravity effects might be relevant, and those could be described by the framework presented in this paper. Still, for each of these cases, one has to deal with the effects of relative locality, which are complementary to curvature of momentum space. We will address these questions in the future publications.

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